Sl_2-quasicontinuous spaces

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4th, July, 2022

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Introduction

The theory of continuous lattices and domains which arose from computer science and logic, is based on the investigation of directed complete posets (dcpos, for short). Since there are important mathematical models which fail to be dcpos, there is an attempt to carry as much as possible of the theory of continuous lattices to as general an ordered structure as possible.

Introduction

Moreover, Erné pointed out the importance of the concept of *standard completions* in the context of generalized continuous posets. In the absence of enough joins, Erné introduced the concept of s_2-*continuous* posets and the *weak Scott topology* by means of the cuts instead of joins. Quasicontinuous domains introduced by Gierz, Lawson and Stralka capture many of the essential features of continuous domains and pop up from time to time generalizing slightly the powerful notion of continuous domains.

Introduction

At the 6th International Symposium in Domain Theory, J.D. Lawson emphasized the need to develop the core of domain theory directly in T_0 topological spaces instead of posets. Moreover, it was pointed out that several results in domain theory can be lifted from the context of posets to T_0 topological spaces.

Introduction

The concept of $\mathcal{S}$-convergence for dcpos was introduced by Scott to characterize continuous domains. It was proved that for a dcpo, the $\mathcal{S}$-convergence is topological if and only if it is a continuous domain. By different convergence, not only are many notions of continuity characterized, but also they make order and topology across each other. Zhao and Ho defined a new way-below relation and a new topology constructed from any given topology on a set using irreducible sets in a T_0 topological space replacing directed subsets and investigated the properties of this derived topology and k-bounded spaces.

Introduction

In this paper, we introduce the concepts of SI_2-quasicontinuous spaces and $\mathcal{GD}$-convergence of nets for arbitrary topological spaces by the cuts. We show that a space is SI_2-quasicontinuous if and only if its weakly irreducible topology is hypercontinuous under inclusion order. Finally we arrive at a conclusion that a T_0 space X is SI_2-quasicontinuous if and only if the $\mathcal{GD}$-convergence in X is topological.

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SI_2-quasicontinuous spaces

Recall some concepts of posets and topological spaces.

For a poset P, $x \in P, A \subseteq P$, define $\downarrow A = \{x \in P : x \leq a$ for some $a \in A\}$ and dually, $\uparrow A$. $A^{\uparrow}$ and $A^{\downarrow}$ denote the sets of all upper and lower bounds of A, respectively. A cut operator δ is defined by $A^{\delta} = (A^{\uparrow})^{\downarrow}$ for all $A \subseteq P$. Notice that whenever A has a join (supremum) then $x \in A^{\delta}$ means $x \leq \vee A$. Let $P^{(<\omega)}$ be the set of all nonempty finite subsets of P.

For a topological space (X, τ), the *specialization order* $\leq$ on X is defined by $y \leq x$ if and only if $y \in \mathrm{cl}(\{x\})$. If (X, τ) is T_0 then the specialization order is a partial order.

SI_2-quasicontinuous spaces

Let (X, τ) be a topological space. A nonempty subset $F \subseteq X$ is called *irreducible* if for every closed sets B and C, whenever $F \subseteq B \cup C$, one has either $F \subseteq B$ or $F \subseteq C$. The set of all irreducible sets of the topological space (X, τ) will be denoted by $\mathrm{Irr}_\tau(X)$ or $\mathrm{Irr}(X)$.

Unless otherwise stated, in the context of T_0 spaces, all order-theoretic concepts such as lower sets, upper sets, etc, are taken with respect to the specialization order of the underlying spaces.

SI_2-quasicontinuous spaces

Lemma

Let (X, τ) be a T_0 space.

(1) *If $D \subseteq X$ is a directed set with respect to the specialization order, then D is irreducible;*

(2) *If $U \subseteq X$ is an open set, then U is an upper set; Similarly, if $F \subseteq X$ is a closed set, then F is a lower set.*

SI_2-quasicontinuous spaces

Definition (see [1])

Let X be a T_0 space and $x \in X$, $A, B \subseteq X$. We say that A is *way below* B and write $A \ll_r B$ if for all irreducible sets $E \subseteq X$, $B \cap E^\delta \neq \varnothing$ implies $A \cap E \neq \varnothing$. We write $A \ll_r x$ for $A \ll_r \{x\}$ and $y \ll_r B$ for $\{y\} \ll_r B$. The set $\{x \in X : A \ll_r x\}$ is denoted by $\Uparrow_r A$.

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[1] S. Z. Luo, X. Q. Xu, On SI_2-continuous Spaces, Electronic Notes in Theoretical Computer Science, **345**(2019), 125-141.

SI_2-quasicontinuous spaces

Definition

A T_0 space (X, τ) is called *SI_2-quasicontinuous* if for all $x \in X$, the following conditions are satisfied:

(1) $w(x) = \{F \subseteq X : F \in X^{(<\omega)}$ and $F \ll_r x\}$ is directed;

(2) $\uparrow x = \bigcap\{\uparrow F : F \in w(x)\}$;

(3) For any $H \in X^{(<\omega)}$, $\Uparrow_r H \in \tau$.

Clearly, an SI_2-continuous space must be SI_2-quasicontinuous, but the converse may not be true.

SI_2-quasicontinuous spaces

Remark

(1) Let P be a poset. Then P is an s_2-quasicontinuous poset if and only if it is an SI_2-quasicontinuous space with respect to the Alexandroff topology.

(2) Let (X, τ) be a T_0 space. If X is an SI_2-quasicontinuous space, then it is also an s_2-quasicontinuous poset under the specialization order. But the converse may not be true.

SI_2-quasicontinuous spaces

Example

Let X be an infinite set with a cofinite topology τ. Then it is a T_1 space. Clearly it is an antichain under the specialization order, and hence it is an s_2-continuous poset. Thus it is also s_2-quasicontinuous. But $\Uparrow_r x = \{x\} \notin \tau$ for all $x \in X$, then (X, τ) is not an SI_2-quasicontinuous space.

Sl_2-quasicontinuous spaces

We now derive the interpolation property of Sl_2-quasicontinuous spaces.

Theorem

Let X be an Sl_2-quasicontinuous space. Then

(1) *For $x \in X$, $F \ll_r x$ implies that there exists $G \in w(x)$ such that $F \ll_r G \ll_r x$.*

(2) *For any $F, H \in X^{(<\omega)}$, $F \ll_r H$ implies that there exists $G \in X^{(<\omega)}$ such that $F \ll_r G \ll_r H$.*

(3) *For all $F \in X^{(<\omega)}$, $E \in Irr(X)$, $F \ll_r E^\delta$ implies that there exists $e \in E$ with $F \ll_r e$.*

SI_2-quasicontinuous spaces

Definition (see [1, 2])

Let (X, τ) be a T_0 space. A subset $U \subseteq X$ is called weakly irreducibly open if the following conditions are satisfied: (1) $U \in \tau$; (2) $F^{\delta} \cap U \neq \emptyset$ implies $F \cap U \neq \emptyset$ for all $F \in \mathrm{Irr}_{\tau}(X)$. The set of all weakly irreducibly open sets of (X, τ) is a topology, which will be called weakly irreducible topology of X and will be denoted by $\tau_{SI_2}(X)$.

[1] S. Z. Luo, X. Q. Xu, On SI_2-continuous Spaces, Electronic Notes in Theoretical Computer Science, **345**(2019), 125-141.

[2] X. J. Ruan, X. Q. Xu , On a new convergence in topological spces, Open Mathematics, **17**(2019), 1716-1723.

SI_2-quasicontinuous spaces

Proposition

Let (X, τ) be an SI_2-quasicontinuous space.

(1) *For any nonempty set $H \subseteq X$, $\Uparrow_r H = int_{\tau_{SI_2(X)}} \uparrow H$.*

(2) *A subset U of X is weakly irreducibly open iff for each $x \in U$ there exists a finite $F \ll_r x$ such that $\uparrow F \subseteq U$.*

(3) *The sets $\{\Uparrow_r F : F \in X^{(<\omega)}\}$ form a basis for $\tau_{SI_2(X)}$.*

SI_2-quasicontinuous spaces

Now we give the topological characterizations of SI_2-quasicontinuous spaces.

Theorem

(1) *X is an SI_2-quasicontinuous space;*

(2) *For all $x \in X$ and $U \in \tau_{SI_2}(X)$, $x \in U$ implies that there exists $F \in X^{(<\omega)}$ such that $x \in int_{\tau_{SI_2}(X)} \uparrow F \subseteq \uparrow F \subseteq U$;*

(3) *$(\tau_{SI_2}(X)), \subseteq)$ is a hypercontinuous lattice.*

$\mathcal{GD}$-convergence in SI_2-quasicontinuous spaces

In this section, the concept of $\mathcal{GD}$-convergence in a poset is introduced. It is proved that the T_0 space X is SI_2-quasicontinuous if and only if the $\mathcal{GD}$-convergence in X is topological.

Definition

Let X be a T_0 space and $(x_j)_{j\in J}$ a net in X. $F \subseteq X$ is called a quasi-eventual lower bound of a net $(x_j)_{j\in J}$ in P, if F is finite and there exists $k \in J$ such that $x_j \in\uparrow F$ for all $k \leq j$ with respect to the specialization order.

Definition

Let X be a T_0 space and $(x_j)_{j\in J}$ a net. We say $(x_j)_{j\in J}$ quasi $\mathcal{D}$ converges to $x \in X$ if there exists a directed family $\mathcal{F} = \{F \subseteq X : F \text{ is finite}\}$ of a quasi-eventual lower bounds of the net $(x_j)_{j\in J}$ in X with respect to the specialization order such that $\bigcap_{F\in\mathcal{F}} \uparrow F \subseteq \uparrow x$. In this case we write $x \equiv_{\mathcal{GD}} \lim x_j$.

$\mathcal{GD}$-convergence in SI_2-quasicontinuous spaces

Proposition

Let X be an SI_2-quasicontinuous space. Then $x \equiv_{\mathcal{GD}} \lim x_j$ if and only if the net $(x_j)_{j\in J}$ converges to the element x with respect to the topology $\tau_{SI_2}(X)$. That is, the $\mathcal{GD}$-convergence is topological.

$\mathcal{GD}$-convergence in Sl_2-quasicontinuous spaces

Proposition

Let (X, τ) be a T_0 space. If the $\mathcal{GD}$-convergence with respect to the topology $\tau_{Sl_2}(X)$ is topological, then X is Sl_2-quasicontinuous.

$\mathcal{GD}$-convergence in SI_2-quasicontinuous spaces

Theorem

Let X be a T_0 space. Then the following statements are equivalent:

(1) *SI_2-quasicontinuous space;*

(2) *The $\mathcal{GD}$-convergence with respect to the topology $\tau_{SI_2}(X)$ is topological, that is, for all $x \in X$ and all nets $(x_j)_{j\in J}$ in X, $x \equiv_{\mathcal{GD}} \lim x_j$ if and only if $(x_j)_{j\in J}$ converges to x with respect to the weakly irreducible topology.*

$\mathcal{GD}$-convergence in SI_2-quasicontinuous spaces

Corollary (see [3])

Let P be a dcpo. Then the following conditions are equivalent:

(1) *P is a quasicontinuous domain;*

(2) *S^*-convergence in P is topological for the Scott topology $\sigma(P)$, that is, for all $x \in P$ and all nets $(x_j)_{j\in J}$ in P, $(x_j)_{j\in J}$ S^*-converges to x if and only if $(x_j)_{j\in J}$ converges to x with respect to the Scott topology.*

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[3] L. J. Zhou, Q. G. Li, Convergence on quasi-continuous domain, Journal of Computational Analysis and Applications, **15**(2013), 381-390.

Corollary (see [2])

Let P be a poset. Then the following conditions are equivalent:

(1) *P is s_2-quasicontinuous;*

(2) *The $\mathcal{GS}$-convergence in P is topological for the weak Scott topology $\sigma_2(P)$, that is, for all $x \in P$ and all nets $(x_j)_{j\in J}$ in P, $x \equiv_{\mathcal{GS}} \lim x_j$ if and only if $(x_j)_{j\in J}$ converges to x with respect to the weak Scott topology.*

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[2] X. J. Ruan, X. Q. Xu, Convergence in s_2-quasicontinuous posets, Springerplus, **4**(2016), 1-10.

Thank you for your attention !